THEORETICAL PREDICTION OF THE EQUILIBRIUM, REAL GAS INVISCID FLOW FIELDS ABOUT BLUNT BODIES

Phase II Formal Written Report

AVCO-EVERETT RESEARCH LABORATORY
a division of
AVCO CORPORATION
Everett, Massachusetts

Contract No. NAS 9-858

June 1963

prepared for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
MANNED SPACECRAFT CENTER
GENERAL RESEARCH PROCUREMENT OFFICE
Houston 1, Texas

I. INTRODUCTION

Several approaches to the blunt body flow problem at zero angle of attack are established theoretically and are in use at the present time. Most of these approaches are summarized in Chapter 6 of Hayes and Probstein[1]. They consist of various approximate techniques for integrating the inviscid equations of flow in the subsonic or subsonic-supersonic region (Figure 1).

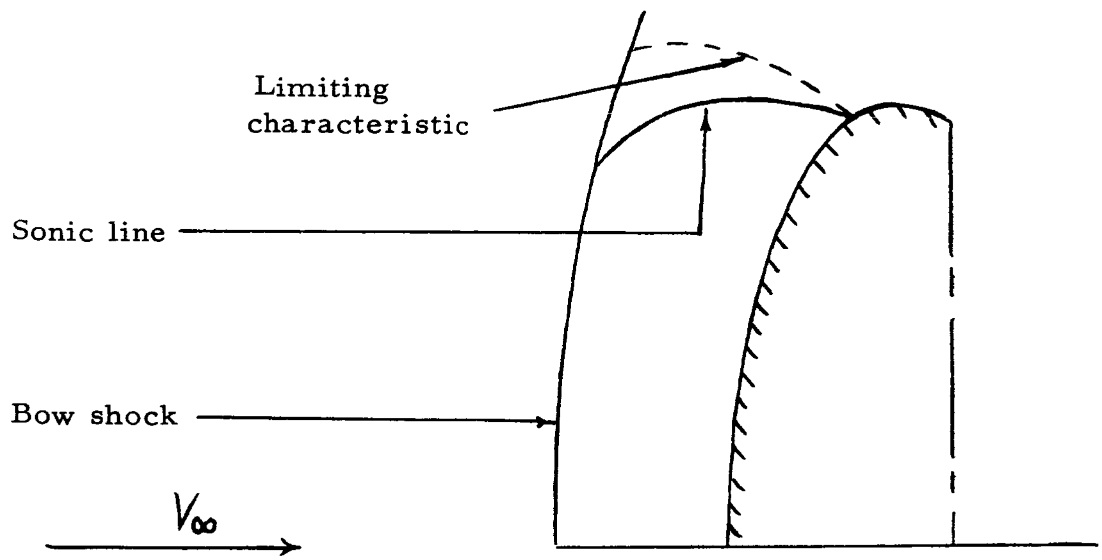

Figure 1. Subsonic-Supersonic Flow Field at Zero Angle of Attack

A complete integration covers the flow field up to the limiting characteristic - that is, the characteristic (or characteristics) farthest downstream which intersects the sonic line - since otherwise, disturbances in the supersonic flow may affect the subsonic region through the sonic line. In order to provide input for method of characteristics calculations, it is necessary to extend the integration somewhat downstream of the limiting characteristic in order to avoid computational difficulties near Mach 1.

Expansion techniques valid near the axis of symmetry and the like are useful but do not give sufficient detail for flow field prediction. The inverse problem has been treated successfully by several authors. In this approach, a shock shape is assumed and the integration is then carried out as if the problem were of the initial value type. Using initial conditions computed at the specified shock, the equations are numerically integrated in some manner until the computed value of the stream function vanishes, say; this locus of points is the resulting body shape. Since such a procedure is not mathematically stable when applied to elliptic (in this case, subsonic) systems of equations, either a smoothing technique must be employed to remove oscillations and irregularities from the solution, or some other method must be used to avoid the instability.

Zlotnick and Newman[2], among several papers[3,4,5] employing the same basic concept, solve the inverse problem by a straightforward finite-difference method. They smooth out unwanted oscillations by systematic fitting of polynomials. Vaglio-Laurin and Ferri[3] also treat the "transonic" region between the sonic line and the limiting characteristic in a consistent manner by integrating along characteristics.

Garabedian and Lieberstein[6,7], an adaptation of whose method appears in section III of this report, introduce a complex transformation which, in a sense, alters the nature of the equations of motion from elliptic to hyperbolic, thereby circumventing the instability problem. The integration from the shock to the unknown body is carried out in the complex domain, and the results are then projected onto the real plane. The stability of the procedure is rooted in the possibility of analytically continuing the functions into the complex domain.

The direct problem, in which the body shape is given and the flow field, including the shock, is solved for, has been treated by Emmons et al.[8] using a streamtube technique. Initial approximations to the shock shape and body pressure distribution are assumed, and the resulting flow field is determined by the streamtube relations. An iterative scheme is employed to correct the shock shape and pressure distribution until the difference between two successive solutions is slight.

The direct solution of Belotserkovskii[9,10] is based on the method of integral relations formulated by Dorodnitsyn[11]. The flow field between the body and the shock is split up into a number of equally spaced strips, and the equations of motion are integrated from the body to each strip boundary. Polynomial expressions are assumed for the dependent va[illegible]bles in the direction normal to the body; upon substitution in the integrated equations, there is obtained a set of ordinary differential equations for the coefficients of the polynomials in terms of the coordinate along the body. These equations are solved with assumed values of the stagnation point standoff distance and the initial velocities at the strip boundaries, and the solution is iterated so as to satisfy properly singularities which occur at the sonic line. Traugott[12] and Holt[13] have presented versions of this method.

All the methods described above can in principle be generalized to the case of a blunt body at angle of attack. A perturbation technique can be formulated for small angles of attack, as has been done by Vaglio-Laurin and Ferri[3] for the finite-difference marching procedure, although they present no results. Swigart[14] perturbs the equations of motion in the small angle of attack and at the same time develops a solution in the form of a series expansion valid near the axis of symmetry of the shock wave (i.e., near the stagnation line). The angle of attack perturbation permits the dependence on the meridional coordinate to be eliminated, and the series expansion results in a set of ordinary differential equations that are integrated inward from the given paraboloidal shock. Swigart's results for zero angle of attack agree well with experimental and other theoretical results for spherical bodies and ellipsoidal bodies of moderate eccentricity, even up to the sonic point; however, his expansion procedure would not be expected to be accurate for blunt bodies with relatively small corner radii, unless many terms in the series were retained.

Swigart observes that his computed body (dividing) streamline at angle of attack differs from the streamline that passes through the point where the shock is normal to the free stream vector -- in other words, the body entropy is not the maximum entropy in the flow field. The question of the body entropy is still a matter of controversy and has not been definitely resolved. Swigart's result may be due to the perturbation scheme he has employed. As pointed out by Ferri[15] in connection with the yawing sharp cone problem, a straightforward perturbation in the angle of attack does not give the proper body entropy distribution.

The papers of Bazzhin[16] and Minailos[17] represent important recent Soviet work in the field of blunt body flow at large angles of attack. Both apply the method of integral relations with one strip in the direction normal to the body, Bazzhin to the two-dimensional flat plate problem and Minailos to the problem of a yawing axisymmetric body. Vaglio-Laurin[18], in addition to his treatment of the application of the PLK method to the calculation of blunt-body flows, presents an analysis similar to Bazzhin's for asymmetric two-dimensional shapes. Both Soviet authors make allowances for the possibility that the dividing streamline does not pass through the normal point of the shock. Bazzhin summarizes the results of his calculations, which were performed with the initial assumption that the dividing streamline did pass through the normal point on the shock. He notes that for angles of attack sufficiently large ($>30^{\circ}$) that his resulting solution indicated that the initial assumption on the dividing streamline was incorrect, it was not possible to iterate the solution in order to obtain successive values of the entropy on the body. He concludes that the one-strip Belotserkovskii-type method does not provide sufficient detail to clarify the problem of body entropy.

A comparison of the various methods developed for zero angle of attack flows, with a view to determining the most suitable approach for the calculation of flows past axisymmetric bodies at large angles of attack, reveals that the inverse methods are conceptually simplest. With the problem of mathematical stability almost completely resolved, there is little difficulty in sketching the broad outlines of an inverse technique, in which the equations of motion are integrated in a step-by-step marching procedure starting from the given shock and working toward the unknown body. A look into the details of the numerical method, however, is rather discouraging. There is scant experience at present in finite-difference methods for three independent variables. A preliminary investigation of the Garabedian and Lieberstein method showed that the problems involved in formulating a method for integrating the equations in the complex domain were much more formidable than expected, and the approach was abandoned. It is also recognized that the selection of the proper shock shape associated with a desired body requires a certain amount of experience, particularly for large angles of attack.

Attention therefore is centered on the direct methods. The streamtube method appears undesirable both from the aspect of its formulation for three dimensions and from certain computational problems caused by the lack of accuracy in computing streamline curvatures numerically. The method of integral relations has the advantage that the variation of properties in the direction normal to the surface is taken into account through expansion in series and integration, so that there remains only the variation in two surface directions. Furthermore, the meridional variation can be approximated by Fourier series. Finally, the method allows a systematic treatment of the sonic surface, and the integration can be extended downstream to provide input for calculations of the supersonic flow.

In section II, there appears a preliminary formulation of the method of integral relations for the problem of large angles of attack. This formulation is restricted to a single strip in the direction normal to the body surface, and the cross-flow term is approximated by assuming a simple sinusoidal variation in the meridional direction.

Section III is concerned with the Garabedian and Lieberstein analysis which has been used for the problem of zero angle of attack.

II. METHOD OF INTEGRAL RELATIONS FOR REAL-GAS FLOW OVER AN AXISYMMETRIC BODY AT LARGE ANGLES OF ATTACK: PRELIMINARY ANALYSIS

A. General Equations

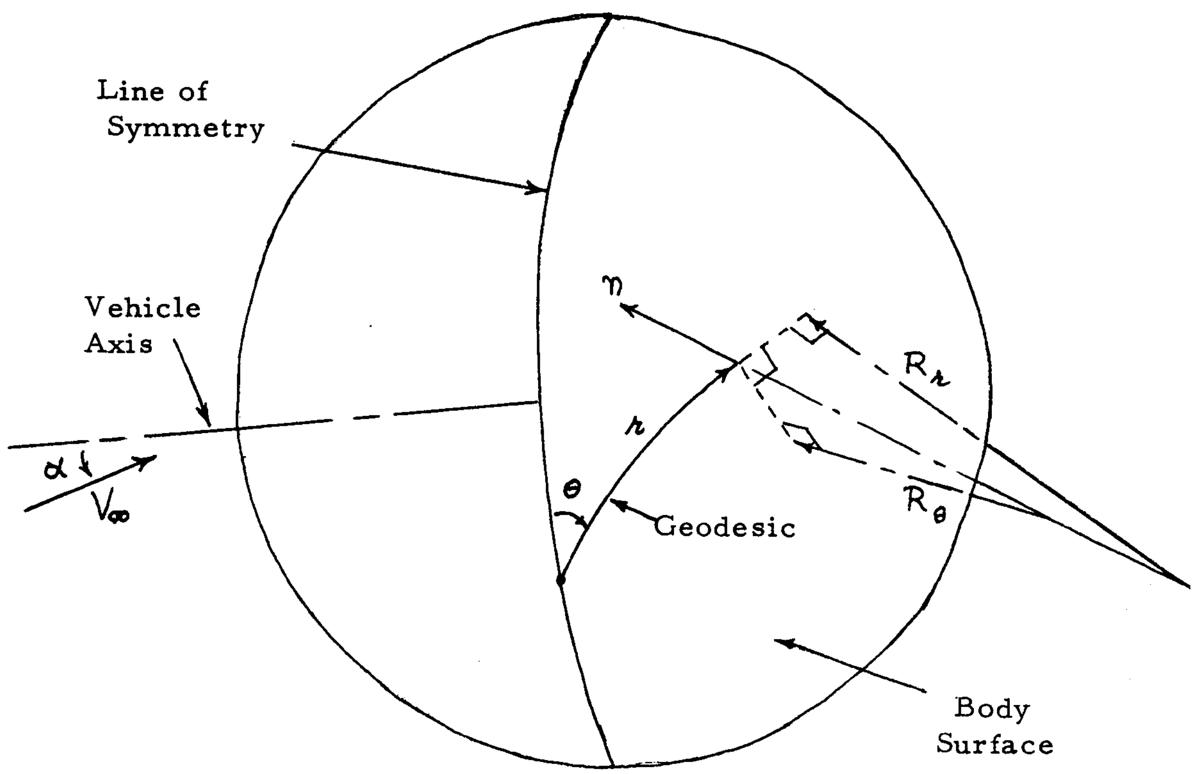

Figure 2. Surface Geometry and Coordinate System

A geodesic, surface-oriented orthogonal coordinate system (s, n, θ) is employed, where n is the distance normal to the body, $n = 0$ is the body surface, s is surface distance along geodesic curves originating at the stagnation point $s = 0$, θ is the angle between the line of symmetry and a geodesic, measured on the body at the stagnation point, $\theta = 0$ is the leeward plane of symmetry, and $\theta = \pi$ is the windward plane of symmetry. Then a general element of length ds is given by

$$ds^2 = h_1^2 \, ds^2 + dn^2 + h_3^2 \, d\theta^2 . \tag{1}$$

Here

$$\begin{aligned} h_1 &= 1 + \frac{n}{R_s} , \\ h_3 &= \left(1 + \frac{n}{R_\theta}\right) \sqrt{G} , \end{aligned} \tag{2}$$

where $R_s\,(s,\theta)$ is the radius of curvature of the surface in the s-direction, $R_\theta\,(s,\theta)$ is the radius of curvature in the θ - direction, and $\sqrt{G(s,\theta)}$ is the surface metric for the θ - coordinate defined, for instance, by Struik[19]. Minailos[17] uses a cylindrical coordinate system with axis along the axis of symmetry of the body; however, it is felt that the probable improvement in convergence achieved with a coordinate system with origin at the stagnation point outweighs the slight additional complication.

In this coordinate system the velocity components in the (s, n, θ) - directions are denoted (u, v, w), respectively. The continuity equation and the equations of momentum in the n- and θ - directions are

$$\frac{\partial}{\partial s}\left(h_3 \rho u\right) + \frac{\partial}{\partial n}\left(h_1 h_3 \rho v\right) + \frac{\partial}{\partial \theta}\left(h_1 \rho w\right) = 0 , \tag{3}$$

$$\frac{u}{h_1}\frac{\partial v}{\partial s} + v \frac{\partial v}{\partial n} + \frac{w}{h_3}\frac{\partial v}{\partial \theta} + \frac{1}{\rho}\frac{\partial p}{\partial n} = \frac{u^2}{h_1}\frac{\partial h_1}{\partial n} + \frac{w^2}{h_3}\frac{\partial h_3}{\partial n} , \tag{4}$$

$$\frac{u}{h_1}\frac{\partial w}{\partial \lambda}+v\frac{\partial w}{\partial n}+\frac{w}{h_3}\frac{\partial w}{\partial \theta}+\frac{1}{h_3\rho}\frac{\partial p}{\partial \theta}=\frac{u}{h_1h_3}\left(u\frac{\partial h_1}{\partial \theta}-w\frac{\partial h_3}{\partial \lambda}\right)-\frac{vw}{h_3}\frac{\partial h_3}{\partial n}\,. \tag{5}$$

By manipulating and making use of equation (3), equations (4) and (5) can be written in the "divergence form"

$$\begin{aligned}\frac{\partial}{\partial \lambda}(h_3\rho u v)+\frac{\partial}{\partial n}\left[h_1h_3(p+\rho v^2)\right]+\frac{\partial}{\partial \theta}(h_1\rho v w)\\ =\rho\left(h_3u^2\frac{\partial h_1}{\partial n}+h_1w^2\frac{\partial h_3}{\partial n}\right)+p\frac{\partial}{\partial n}(h_1h_3),\end{aligned} \tag{6}$$

$$\begin{aligned}\frac{\partial}{\partial \lambda}(h_3\rho u w)+\frac{\partial}{\partial n}(h_1h_3\rho v w)+\frac{\partial}{\partial \theta}\left[h_1(p+\rho w^2)\right]\\ =\rho\left(u^2\frac{\partial h_1}{\partial \theta}-uw\frac{\partial h_3}{\partial \lambda}-h_1vw\frac{\partial h_3}{\partial n}\right)+p\frac{\partial h_1}{\partial \theta}\,.\end{aligned} \tag{7}$$

Before performing the integration in the n - direction, the variable n is replaced by the dimensionless variable $\zeta = n/\epsilon$, where ϵ is the value of n at the shock wave. In terms of (λ, ζ, θ) , equations (3), (6), and (7) become, respectively,

$$\frac{\partial P_i}{\partial \lambda}+\frac{\partial Q_i}{\partial \zeta}+\frac{\partial R_i}{\partial \theta}=L_i \qquad (i=1,2,3), \tag{8}$$

where

$$\begin{aligned}&P_1=h_3\rho u\,,\quad P_2=h_3\rho u v\,,\quad P_3=h_3\rho u w,\\ &Q_1=\frac{\rho}{\epsilon}\left[h_1h_3v-\zeta\left(h_3u\frac{\partial \epsilon}{\partial \lambda}+h_1w\frac{\partial \epsilon}{\partial \theta}\right)\right]=\zeta\rho L_1+\frac{\rho h_1h_3v}{\epsilon},\\ &Q_2=vQ_1+\frac{h_1h_3p}{\epsilon}\,,\qquad Q_3=wQ_1-\frac{\zeta h_1p}{\epsilon}\frac{\partial \epsilon}{\partial \theta},\\ &L_1=-\frac{\rho}{\epsilon}\left(h_3u\frac{\partial \epsilon}{\partial \lambda}+h_1w\frac{\partial \epsilon}{\partial \theta}\right),\\ &L_2=vL_1+\rho\left(h_3u^2\frac{\partial h_1}{\partial n}+h_1w^2\frac{\partial h_3}{\partial n}\right)+p\frac{\partial}{\partial n}(h_1h_3),\\ &L_3=wL_1+\rho\left(u^2\frac{\partial h_1}{\partial \theta}-uw\frac{\partial h_3}{\partial \lambda}-h_1vw\frac{\partial h_3}{\partial n}\right)+\epsilon p\frac{\partial}{\partial \theta}\left(\frac{h_1}{\epsilon}\right).\end{aligned} \tag{9}$$

Equations (8) are now integrated with respect to ζ from 0 (body) to 1 (shock), and the coefficients P_i, R_i, L_i are assumed to have linear profiles in ζ :

$$P_i = (1-\zeta)\, P_{i_b} + \zeta\, P_{i_s} \quad , \text{etc.,} \tag{10}$$

where P_{i_b} is the value of P_i at the body and P_{i_s} is the value of P_i at the shock. The results of these integrations are

$$\frac{\partial P_{i_b}}{\partial \lambda} + \frac{\partial P_{i_s}}{\partial \lambda} = 2\,(Q_{i_b} - Q_{i_s}) + L_{i_b} + L_{i_s} - \frac{\partial R_{i_b}}{\partial \theta} - \frac{\partial R_{i_s}}{\partial \theta} . \tag{11}$$

The equation of conservation of stagnation enthalpy

$$h + \tfrac{1}{2}\,(u^2 + v^2 + w^2) = H \tag{12}$$

becomes at the body $(v_b = 0)$ $\quad h_b + \frac{1}{2}\,(u_b^{\,2} + w_b^{\,2}) = H,$

or $\quad \dfrac{\partial h_b}{\partial \lambda} = -\left(u\,\dfrac{\partial u}{\partial \lambda} + w\,\dfrac{\partial w}{\partial \lambda}\right)_b .$

Since the entropy is constant on the body,

$$\frac{\partial \rho_b}{\partial \lambda} = \left(\frac{\partial \rho}{\partial h}\right)_S \frac{\partial h_b}{\partial \lambda} = -\left[\frac{\rho}{a^2}\left(u\,\frac{\partial u}{\partial \lambda} + w\,\frac{\partial w}{\partial \lambda}\right)\right]_b . \tag{13}$$

Equation (13) and the first and third of equation (11) can be solved for $\frac{\partial u_b}{\partial \lambda}$ by eliminating $\frac{\partial \rho_b}{\partial \lambda}$ and $\frac{\partial w_b}{\partial \lambda}$, resulting in the fundamental equation

$$\left[h_3 \rho \left(1 - \frac{u^2}{a^2}\right)\frac{\partial u}{\partial \lambda}\right]_b = \left(1 - \frac{w^2}{a^2}\right)_b A + \left(\frac{w}{a^2}\right)_b B, \tag{14}$$

where

$$A = -\left[\frac{\rho u}{\epsilon} \frac{\partial}{\partial n} (h_3 \epsilon) + \frac{1}{\epsilon} \frac{\partial}{\partial \theta} (h_1 \epsilon \rho w) \right]_b \tag{15}$$
$$- \left[\epsilon \frac{\partial}{\partial n} \left(\frac{h_3 \rho u}{\epsilon} \right) + \epsilon \frac{\partial}{\partial \theta} \left(\frac{h_1 \rho w}{\epsilon} \right) + \frac{2 h_1 h_3 \rho v}{\epsilon} \right]_s ,$$

$$B = -\left[\frac{1}{\epsilon} \frac{\partial}{\partial \theta} (h_1 \epsilon \rho w^2) + \rho u \left(u \frac{\partial h_1}{\partial \theta} + \frac{h_3 w}{\epsilon} \frac{\partial \epsilon}{\partial n} \right) - \frac{h_1}{\epsilon} \frac{\partial}{\partial \theta} (\epsilon p) \right]_b$$
$$- \left[\epsilon \frac{\partial}{\partial n} \left(\frac{h_3 \rho u w}{\epsilon} \right) + \epsilon \frac{\partial}{\partial \theta} \left(\frac{h_1 \rho w^2}{\epsilon} \right) - \rho \left\{ u^2 \frac{\partial h_1}{\partial \theta} - u w \frac{\partial h_3}{\partial n} \right. \right. \tag{16}$$
$$\left. \left. - h_1 v w \left(\frac{\partial h_3}{\partial n} + \frac{2 h_3}{\epsilon} \right) \right\} + h_1 \epsilon \frac{\partial}{\partial \theta} \left(\frac{p}{\epsilon} \right) \right]_s .$$

The second of equation (11) yields

$$\left[\epsilon \frac{\partial}{\partial n} \left(\frac{h_3 \rho u v}{\epsilon} \right) \right]_s = \left[\rho \left(h_3 u^2 \frac{\partial h_1}{\partial n} + h_1 w^2 \frac{\partial h_3}{\partial n} \right) \right.$$
$$\left. + p \left\{ \frac{\partial}{\partial n} (h_1 h_3) + \frac{2 h_1 h_3}{\epsilon} \right\} \right]_b + \left[- \epsilon \frac{\partial}{\partial \theta} \left(\frac{h_1 \rho v w}{\epsilon} \right) \right. \tag{17}$$
$$\left. + \rho \left(h_3 u^2 \frac{\partial h_1}{\partial n} - \frac{2 h_1 h_3 v^2}{\epsilon} + h_1 w^2 \frac{\partial h_3}{\partial n} \right) + p \left\{ \frac{\partial}{\partial n} (h_1 h_3) - \frac{2 h_1 h_3}{\epsilon} \right\} \right]_s .$$

Equation (17) governs the variation of the shock quantities. The general shock relations are written in terms of the velocity components $V_{N\infty}$ (free stream) and V_N (behind the shock) normal to the shock wave. The conditions of continuity, normal momentum conservation, and energy conservation are.

$$\rho_s = \rho_\infty \frac{V_{N\infty}}{V_N} , \tag{18}$$

$$p_s = p_\infty + \rho_\infty V_{N\infty} \left(V_{N\infty} - V_N \right) , \tag{19}$$

$$h_s = h_\infty + \frac{1}{2} \left(V_{N\infty}{}^2 - V_N{}^2 \right) . \tag{20}$$

The condition of tangential momentum conservation yields the relations

$$u_s = u_\infty + (v_\infty - v_s)\frac{1}{h_1}\frac{\partial\epsilon}{\partial n}, \tag{21}$$

$$w_s = w_\infty + (v_\infty - v_s)\frac{1}{h_3}\frac{\partial\epsilon}{\partial n}, \tag{22}$$

while the velocity component v_s can be related to the normal velocity component and the shock geometry by

$$v_s = -V_N\left[1+\left(\frac{1}{h_1}\frac{\partial\epsilon}{\partial n}\right)^2+\left(\frac{1}{h_3}\frac{\partial\epsilon}{\partial\theta}\right)^2\right]^{-1/2}$$
$$+\left[\left(u_\infty+\frac{v_\infty}{h_1}\frac{\partial\epsilon}{\partial n}\right)\frac{1}{h_1}\frac{\partial\epsilon}{\partial n}+\left(w_\infty+\frac{v_\infty}{h_3}\frac{\partial\epsilon}{\partial\theta}\right)\frac{1}{h_3}\frac{\partial\epsilon}{\partial\theta}\right]\left[1+\left(\frac{1}{h_1}\frac{\partial\epsilon}{\partial n}\right)^2+\left(\frac{1}{h_3}\frac{\partial\epsilon}{\partial\theta}\right)^2\right]^{-1}. \tag{23}$$

Equations (18)-(23) essentially render the shock variables as functions of the shock geometry. An additional relation, the equation of state in terms of ρ, p, and h, is needed to complete the system. The system is solved in an iterative manner for equilibrium air, whose equation of state in this instance is in the form of curve fits, by requiring compatibility of equations (18)-(20) with the equation of state.

B. Method of Solution for a Sinusoidal Distribution in θ

In the preceding analysis, the only assumption has been that the profiles in the n-direction are linear. The relevant equations are now written in the planes $\theta = 0, \pi$, where by symmetry $\frac{\partial p}{\partial\theta} = \frac{\partial\rho}{\partial\theta} = \cdots = 0$, $w = 0$, $\frac{\partial w}{\partial\theta} \neq 0$. Equation (14) becomes

$$\left[h_3\rho\left(1-\frac{u^2}{a^2}\right)\frac{du}{dn}\right]_b = -\left[\frac{\rho u}{\epsilon}\frac{d}{dn}(h_3\epsilon)+h_1\rho\frac{\partial w}{\partial\theta}\right]_b$$
$$-\left[\epsilon\frac{d}{dn}\left(\frac{h_3\rho u}{\epsilon}\right)+h_1\rho\frac{\partial w}{\partial\theta}+\frac{2h_1h_3\rho v}{\epsilon}\right]_s, \tag{24}$$

while equation (17) becomes

$$\left[\epsilon \frac{d}{ds}\left(\frac{h_3 \rho u v}{\epsilon}\right)\right]_s = \left[h_3 \rho u^2 \frac{\partial h_1}{\partial n} + p\left\{\frac{\partial}{\partial n}(h_1 h_3) + \frac{2h_1 h_3}{\epsilon}\right\}\right]_b$$

$$+ \left[-h_1 \rho v \frac{\partial w}{\partial \theta} + \rho\left(h_3 u^2 \frac{\partial h_1}{\partial n} - \frac{2h_1 h_3 v^2}{\epsilon}\right)\right. \tag{25}$$

$$\left. + p\left\{\frac{\partial}{\partial n}(h_1 h_3) - \frac{2h_1 h_3}{\epsilon}\right\}\right]_s .$$

The shock conditions (18)-(20) remain unchanged, while equations (21) and (23) become

$$u_s = V_N \sin\lambda + V_\infty \cos\lambda \cos(\lambda + \delta \pm \alpha), \tag{26}$$

$$v_s = -V_N \cos\lambda + V_\infty \sin\lambda \cos(\lambda + \delta \pm \alpha), \tag{27}$$

with $V_{N_\infty} = V_\infty \sin(\lambda + \delta \pm \alpha)$. Here the plus sign is taken on the windward side of the body axis of symmetry, the minus sign on the leeward side. The shock geometry and the angles λ and δ are illustrated in Figure 3. The shock conditions are now considered to be functions of the shock angle λ defined by

$$\frac{1}{h_{1_s}} \frac{d\epsilon}{ds} = \tan\lambda . \tag{28}$$

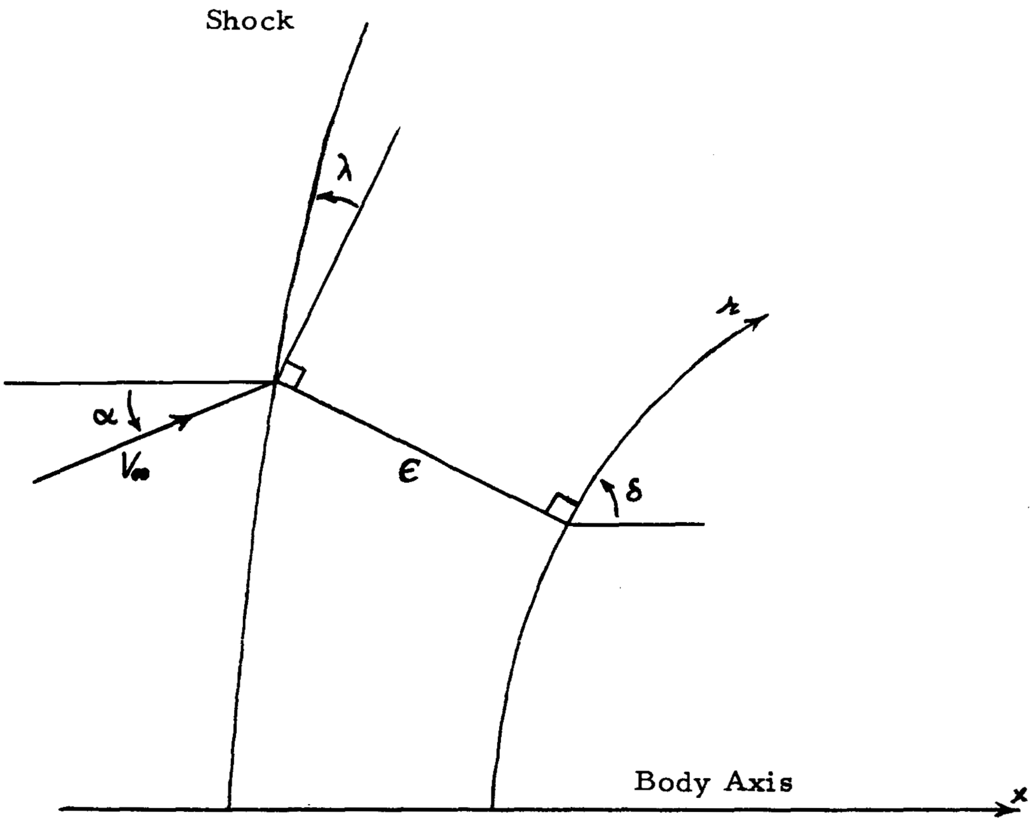

Figure 3. Shock Wave Geometry

The radius of curvature $R_\lambda(\lambda)$ in equation (2) is the radius of curvature of the body contour in the plane of symmetry, while R_θ is the distance from the body to the body axis measured along the normal to the body. It is most convenient to define an auxiliary cylindrical coordinate system with x - axis along the body axis and y - axis $(y \geq 0)$ in the plane of symmetry, as indicated in Figure 4. Then

$$R_\theta = \frac{y}{\cos \delta} \tag{29}$$

($\delta \leq \frac{\pi}{2}$ everywhere).

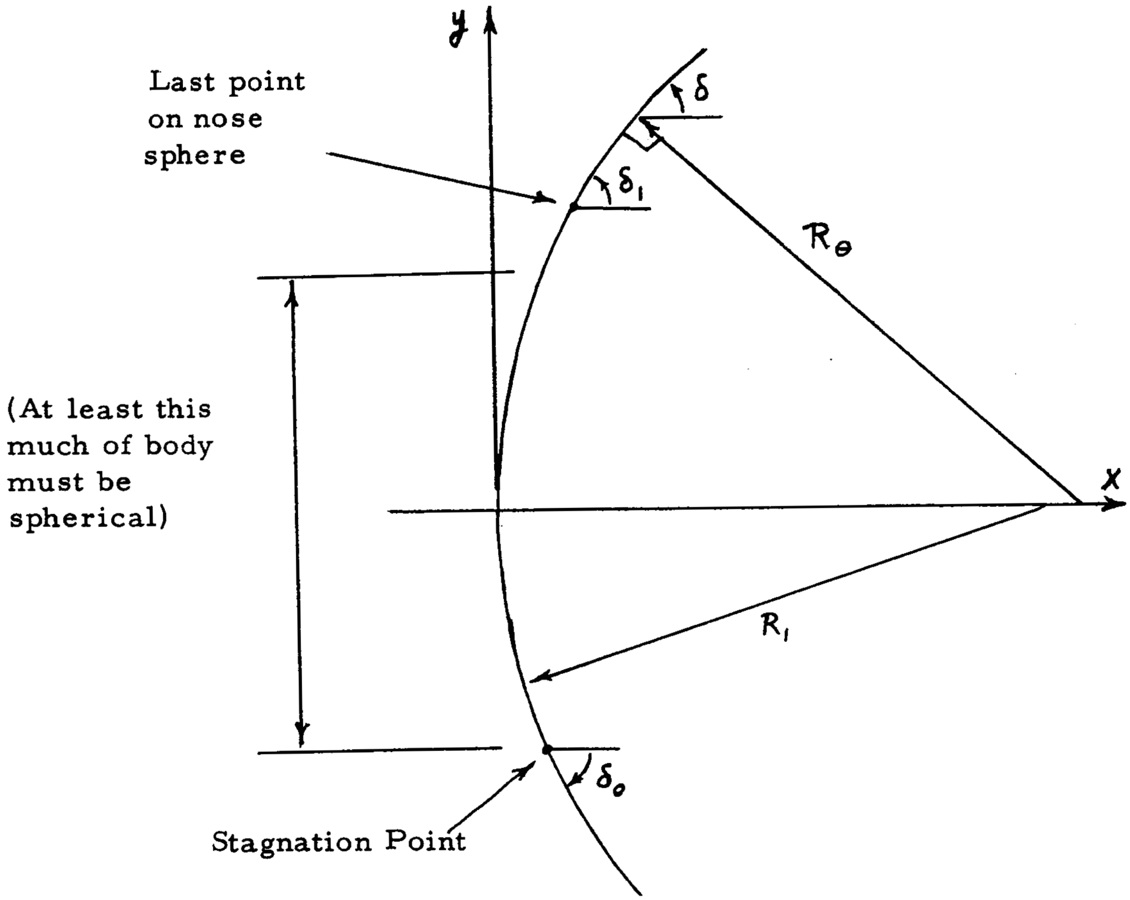

Figure 4. Auxiliary Cylindrical Coordinate System

The metric $\sqrt{G(n)}$ is easiest to evaluate if it is assumed that the body contour is spherical at least down to the stagnation point, with radius R_1. Let the slope of the body at the last point on the sphere be δ_1. The value of the metric on the nose sphere is

$$\sqrt{G} = R_1 \sin(\delta \mp \delta_0) \quad \left\{ \begin{array}{l} - \text{ below body axis} \\ + \text{ above body axis} \end{array} \right. \tag{30}$$

while its value on the rest of the body is

$$\sqrt{G} = \frac{y \sin(\delta_1 \mp \delta_0)}{\cos\delta_1} + y R_1 \cos\delta_0 \int_{R_1\cos\delta_1}^{y} \frac{dy}{y^2 \sin\delta} \quad . \tag{31}$$

Of course it would be possible to evaluate the metric for other configurations.

It remains to evaluate the cross-flow term $\frac{\partial w}{\partial\theta}$ appearing in the integrated equations (24) and (25). This is done by assuming that the variation of the cross-flow velocity component in the circumferential direction is given with adequate accuracy by

$$w = w^{\pi/2} \sin\theta ,$$

while the variation of the other flow quantities (enthalpy, u, v) is given by

$$h = \frac{h^0 + h^\pi}{2} + \frac{h^0 - h^\pi}{2} \cos\theta ,$$

etc. Here the superscripts 0, $\frac{\pi}{2}$, π refer to the values of θ at which the functions are evaluated. In the surface $\theta = \frac{\pi}{2}$, the equation of conservation of stagnation enthalpy (12) gives

$$h^{\pi/2} + \frac{1}{2}\left[(u^{\pi/2})^2 + (v^{\pi/2})^2 + (w^{\pi/2})^2 \right] = H ,$$

or, solving formally for $w^{\pi/2}$,

$$w^{\pi/2} = \left[2H - h^0 - h^\pi - \left(\frac{u^0 + u^\pi}{2}\right)^2 - \left(\frac{v^0 + v^\pi}{2}\right)^2 \right]^{1/2} . \tag{32}$$

The quantity $\frac{\partial w}{\partial\theta}$ in the planes $\theta = 0, \pi$ is then determined from the equation

$$\frac{\partial w}{\partial\theta} = w^{\pi/2} \cos\theta \tag{33}$$

evaluated at either the shock or the body.

The differential equations (24), (25), and (28), together with the shock conditions (18)-(20), (26), and (27), and equations (32) and (33) for $\partial v/\partial\theta$, are sufficient to determine the solution point-by-point (for successive values of r) in the planes $\theta = 0, \pi$. The procedure to determine an unknown point, given the solution at one or more previous points, is as follows:

1. The right-hand side of equation (25) is evaluated at the previous point to determine $\frac{d}{dr}\left(\frac{h_3 \rho u v}{\epsilon}\right)_s$, and this derivative is multiplied by the change in r between the points and added to the previous value of $\left(\frac{h_3 \rho u v}{\epsilon}\right)_s$, yielding a first approximation to $\left(\frac{\rho u v}{\epsilon}\right)_s$ at the unknown point; or a more accurate numerical procedure is used. The same is done to equation (28) to determine a first approximation to ϵ.

2. An iterative scheme is developed to determine that value of λ at the unknown point which yields the computed value of $(\rho u v)_s$, using the shock conditions. This step replaces the tedious differentiation of the shock relations in other treatments of the method of integral relations.

3. The right-hand side of equation (24) is evaluated at the previous point, as in step 1, to yield a first approximation to u_b at the unknown point.

4. Using this value of u_b, the body enthalpy is computed from

$$h_b = H - \frac{1}{2} u_b^2 . \qquad (34)$$

The constant value of the entropy on the body together with the equation of state then yield the values of the other thermodynamic variables.

5. A more accurate solution is obtained by an iterative procedure, using standard numerical techniques.

The computation outlined above must be performed simultaneously in the windward ($\theta = \pi$) and leeward ($\theta = 0$) planes so that equation (32) can be evaluated. The equations are thus integrated in both directions away from the assumed stagnation point. At the two sonic points on the body, equation (24) is singular and cannot be solved unless the right-hand side vanishes along with the left. Hence the sonic points represent natural boundaries of the solution, although it can of course be extended through them with proper choice of the starting conditions.

There are four unknown parameters to be determined by the solution: the location of the stagnation point (δ_o), the shock standoff distance ϵ_o and slope λ_o opposite the stagnation point, and the value of the entropy on the body S_o . Two of these parameters are essentially determined by satisfying the two sonic point conditions. Vaglio-Laurin[18] and Bazzhin[16] make use of a mass-continuity relation as a third condition, and Bazzhin makes use of a fourth condition on S_o without success. R. F. Probstein* has pointed out that neither of the last two conditions may be appropriate, and a study is presently being made of this subject. Results of the study will be reported later. In the meantime it will be assumed that appropriate conditions can be derived to specify the solution uniquely.

*Private communication.

The complete integration is iterated according to the following plan:

1. Values of two of the unknown parameters δ_o, ϵ_o, λ_o, and S_o are guessed. The solution is developed until the sonic points are approached, and the value of the right-hand side of equation (24) is then obtained by extrapolation.

2. Two alternative solutions are developed by respective variation of the unknown parameters. The change in the right-hand side of equation (24) at the sonic points corresponding to each variation is noted.

3. New values of the unknown parameters are chosen to minimize the right-hand side of equation (24), based on the information obtained in steps 1 and 2.

4. The procedure is iterated until a satisfactory solution is reached.

C. Future Work

Future work on this task will be devoted to clarifying the details of the one-strip sinusoidal analysis, programming this analysis and interpreting the results, and generalizing the method to provide a more accurate solution.

III. FLOW FIELD DETERMINATION IN THE SUBSONIC REGION AT ZERO ANGLE OF ATTACK

A more complete report on this topic will be published in the near future. In the meantime, this short description of the mathematical basis of the approach of Garabedian and Lieberstein[6,7] will serve as an introduction to the subject.

The governing differential equation of flow in terms of the stream function ψ is

$$A\frac{\partial^2\psi}{\partial x^2} + 2B\frac{\partial^2\psi}{\partial x\partial y} + C\frac{\partial^2\psi}{\partial y^2} + D = 0, \tag{35}$$

where

$$\begin{aligned} A &= 1-\frac{u^2}{a^2}, \quad B = -\frac{uv}{a^2}, \quad C = 1-\frac{v^2}{a^2}, \\ D &= y^2\rho^2\left(\frac{q^2+h}{\gamma}\right)\frac{d}{d\psi}(\ln E) - \rho u, \\ \psi_x &= -y\rho v, \quad \psi_y = y\rho u, \\ q^2 &= u^2+v^2, \\ E &= \frac{p}{\rho^\gamma}, \end{aligned} \tag{36}$$

and (u, v) are the axial and radial velocity components corresponding to the coordinates (x, y).

If we set

$$P = \frac{\partial \psi}{\partial x}, \quad Q = \frac{\partial \psi}{\partial y}, \qquad (37)$$
$$\Delta = \sqrt{AC - B^2},$$

then equation (35) can be put into the following form:

$$\frac{\partial y}{\partial t} = \frac{1}{\Delta}\left(C \frac{\partial x}{\partial s} - B \frac{\partial y}{\partial s}\right),$$
$$\frac{\partial x}{\partial t} = \frac{1}{\Delta}\left(B \frac{\partial x}{\partial s} - A \frac{\partial y}{\partial s}\right),$$
$$\frac{\partial P}{\partial t} = \frac{1}{\Delta}\left(B \frac{\partial P}{\partial s} + C \frac{\partial Q}{\partial s} + D \frac{\partial x}{\partial s}\right), \qquad (38)$$
$$\frac{\partial Q}{\partial t} = -\frac{1}{\Delta}\left(A \frac{\partial P}{\partial s} + B \frac{\partial Q}{\partial s} + D \frac{\partial x}{\partial s}\right),$$
$$\frac{\partial \psi}{\partial t} = (PB + QC) \frac{\partial x}{\partial s} - (PA + QB) \frac{\partial y}{\partial s},$$

where $\alpha = t + is$, $\beta = t - is$ are the characteristic coordinates of the system.

We augment these equations with the following set of initial conditions, which are taken at the shock:

$$x = f_1(y),$$
$$P = f_2(y),$$
$$Q = f_3(y), \qquad (39)$$
$$\psi = f_4(y),$$

where $f_1, \ldots, f_4$ must be derived for an assumed analytic shock shape and are not given here. The set represented by (39) yields explicit information on the dependent variables as functions of y.

We now set

$$s = \xi + i\eta \tag{40}$$

and keep t real. Specifically, we keep ξ as a parameter and η as the independent variable. This extension into the complex domain is completed when at $t = 0$, the initial values are also extended into the complex domain in the same manner as already described. Physically relevant flow quantities are obtained whenever $\eta = 0$. The body point is obtained whenever $\psi = 0$. This initial value problem is repeated for different values of the parameter ξ so that the entire subsonic flow field can be mapped out.

The method described above has been programmed for digital computation. Its extension from a perfect gas to equilibrium air is largely a matter of changing the coefficient D, and extending the Mollier diagram curve fits into the complex domain by substitution in the curve fit equations. This extension has been formulated and recently programmed, and will be the major subject of the forthcoming report.

<u>References</u>

1. Hayes, W. and Probstein, R., <u>Hypersonic Flow Theory</u>. Academic Press, 1959.

2. Zlotnick, W. and Newman, D., "Theoretical Calculation of the Flow on Blunt-Noxed Axisymmetric Bodies in a Hypersonic Stream", Avco RAD TR-2-57-29, 1957.

3. Vaglio-Laurin, R. and Ferri, A., "Theoretical Investigation of the Flow Field About Blunt-Nosed Bodies in Supersonic Flight", J. Aerospace Sci., Vol. 25, 761, 1958.

4. Van Dyke, W., "The Supersonic Blunt-Body Problem - Review and Extension", J. Aerospace Sci., Vol. 25, p. 485, 1958.

5. Mangler, K., "The Calculation of the Flow Field Between a Blunt Body and the Bow Wave", in <u>Hypersonic Flow</u> (Collan and Tinkler, eds.), Academic Press, 1960.

6. Garabedian, P., "Numerical Construction of Detached Shock Waves", J. Math. Phys., Vol. 36, p. 192, 1957.

7. Garabedian, P. and Lieberstein, H., "On the Numerical Calculation of Detached Bow Shock Waves in Hypersonic Flow", J. Aero. Sci., Vol. 25, p. 109, 1958.

8. Gravalos, F., Edelfelt, I., and Emmons, H., "The Supersonic Flow About a Blunt Body of Revolution for Gases at Chemical Equilibrium', 9th Annual Congress, Int'l Astronautical Fed., 1958.

9. Belotserkovskii, O., "Flow Past a Circular Cylinder with a Detached Shock Wave", Vychislitel'naia Matematika, Vol. 3, p. 149, 1958 (translated by S. Adashko, ed. by M. Holt, Avco RAD-9-TM-59-66, 1959).

10. Belotserkovskii, O., "The Calculation of Flow over Axisymmetric Bodies with a Decaying Shock Wave", Academy of Sciences, USSR Computation Center Manograph, 1961 (translated by J. F. Springfield, Avco RAD-TM-62-64, 1962).

11. Dorodnitsyn, A., "On a Method for the Numerical Solution of Certain Problems of Aero-Hydrodynamics", Academy of Sciences, USSR, Proceedings of 3rd All-Soviet Math. Congress, Vol.2 (1956), Vol. 3 (1958).

12. Traugott, S., "An Approximate Solution of the Direct Supersonic Blunt-Body Problem for Arbitrary Axisymmetric Shapes", J. Aerospace Sci., Vol. 27, No. 5, p. 361, 1960.

13. Holt, M., "Direct Calculation of Pressure Distribution on Blunt Hypersonic Nose Shapes with Sharp Corners", J. Aerospace Sci., Vol. 28, No. 11, p. 872, 1961.

14. Swigart, R., "A Theory of Asymmetric Hypersonic Blunt-Body Flows", AIAA Journal, Vol. 1, No. 5, p. 1034, 1963.

15. Ferri, A., "Supersonic Flow Around Circular Cones at Angles of Attack", NACA Report 1045, 1951.

16. Bazzhin, A., "The Calculation of Supersonic Flow Past a Flat Plate with a Detached Shock Wave", (translated by R. F. Probstein, Avco RAD TM to be published).

17. Minailos, A., "On the Calculation of Supersonic Flow Past Blunted Bodies of Revolution at Angles of Attack", (translated by R. F. Probstein, Avco RAD TM to be published).

18. Vaglio-Laurin, R., "On the PLKMethod and the Supersonic Blunt-Body Problem", J. Aerospace Sci., Vol. 29, No. 2, p. 185, 1962.

19. Struick, D., Lectures on Classical Differential Geometry, Addison-Wesley, 1950.